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J. Math. Anal. Appl. 337 (2008) 174–196

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Spectrum of a network of Euler–Bernoulli beams

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Received 5 October 2006

Available online 3 April 2007

Submitted by Goong Chen

Abstract

A network of N flexible beams connected by n vibrating point masses is considered. The spectrum of the spatial operator involved in this evolution problem is studied. If λ^2 is any real number outside a discrete set of values S and if λ is an eigenvalue, then it satisfies a characteristic equation which is given. The associated eigenvectors are also characterized. If λ^2 lies in S and if the N beams are identical (same mechanical properties), another characteristic equation is available. It is not the case for different beams: no general result can be stated. Some numerical examples and counterexamples are given to illustrate the impossibility of such a generalization. At last the asymptotic behaviour of the eigenvalues is investigated by proving the so-called Weyl's formula.

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Keywords: Network; Flexible beams; Point masses; Spectrum; Characteristic equation; Asymptotics

1. Introduction

In the last few years various physical models of multi-link flexible structures consisting of finitely many interconnected flexible elements such as strings, beams, plates, shells have been mathematically studied. See [12,13,18,25,27] for instance. The spectral analysis of such structures has some applications to control or stabilization problems ([25] and [26]). For interconnected strings (corresponding to a second-order operator on each string), a lot of results have been obtained: the asymptotic behaviour of the eigenvalues [1,2,11,31], the relationship between

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the eigenvalues and algebraic theory (cf. [8,9,25,30]), qualitative properties of solutions (see [11] and [33]) and finally studies of the Green function (cf. [23,34,36]).

For interconnected beams (corresponding to a fourth-order operator on each beam), some results on the asymptotic behaviour of the eigenvalues and on the relationship between the eigenvalues and algebraic theory were obtained by Nicaise and Dekoninck in [20,21] and [22] with different kinds of connections using the method developed by von Below in [8] to get the characteristic equation associated to the eigenvalues.

The same approach will be used in this paper to find the spectrum but with a hybrid system of N flexible beams connected by n vibrating point masses. This type of structure was studied by Castro and Zuazua in many papers (see [14–17,19]) and Castro and Hansen [24]. They have restricted themselves to the case of two beams applying their results on the spectral theory to controllability. They have shown that if the constant of rotational inertia is positive, due to the presence of the mass, the system is well-posed in asymmetric spaces (spaces with different regularity on both sides of the mass) and consequently, the space of controllable data is also asymmetric. For a vanishing constant of rotational inertia the system is not well-posed in asymmetric spaces and the presence of the point mass does not affect the controllability of the system.

Note that S.W. Taylor proved similar results at the same time in [37] using different techniques based on the method presented in [28] for exact controllability.

We will investigate the more general situation of N beams but only compute the spectrum since this case is more complicated to deal with. Namely, on a finite network made of N edges k_j with length l_j , $j = 1, \dots, N$, we consider the eigenvalue problem:

$$\begin{cases} a_j u_{jx_j^{(4)}} = \lambda^2 u_j & \text{on } (0, l_j), \forall j \in \{1, \dots, N\}, \\ \sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v^3}(E_i) = \lambda^2 M_i z_i, & \forall i \in I_{\text{int}}, \\ u_j \in H^4((0, l_j)), & \forall j \in \{1, \dots, n\}, \end{cases}$$

where a_j is a strictly positive mechanical constant. The beams are connected through some conditions on the u_j 's and their first and second order derivatives at the nodes (see Section 2.2).

We establish that, if λ^2 is any real number outside a discrete set of values S and if λ^2 is an eigenvalue, then it satisfies a transcendental equation of the form

$$\det \mathcal{D}(\sqrt{\lambda}) = 0.$$

The associated eigenvectors are also characterized (see Theorem 6, Section 3.2.1).

If λ^2 lies in S and if the N beams are identical (same mechanical properties), another characteristic equation is available (cf. Theorem 8, Section 3.2.1).

All our results can be used directly for numerical applications to determine the eigenelements.

Now in the case of different beams, no general result can be stated. Some numerical examples and counterexamples are given to illustrate the impossibility of such a generalization (see Section 3.2.2).

The case of a chain of $N = 3$ identical beams is treated: additional eigenvalues appear compared to the case of $N = 2$ beams studied by Castro and Zuazua [17] but the asymptotic behaviour of the spectral gap does not change.

To finish with, the asymptotic behaviour of the eigenvalues is presented in Section 4. Following von Below [11] as Ali Mehmeti and Nicaise have done before ([1,31] and [22]), we establish

the Weyl's formula with the help of the min–max principle of Courant–Weyl: if $\{\mu_k\}_{k \in \mathbb{N}}$ denotes the set of eigenvalues of the above eigenvalue problem in increasing order, then

$$\lim_{k \rightarrow \infty} \frac{\mu_k}{k^4} = \pi^4 \left(\sum_{j=1}^N l_j a_j^{-1/4} \right)^{-4}.$$

Before starting the spectral analysis of Section 3, we recall in Section 2 the terminology of networks as they can be found in early contributions of Lumer and Gramsch as well as in papers by Ali Mehmeti ([4] and [5]), von Below [8] and Nicaise ([3,7] and [30]) in the eighties. The authors have also been working on transmission problems on networks for a few years: Mercier studied in [29] transmission problems for elliptic systems in the sense of Agmon–Douglis–Nirenberg on polygonal networks with general boundary and interface conditions.

In [6], Régnier and Ali Mehmeti studied the spectral solution of a one-dimensional Klein–Gordon transmission problem corresponding to a particle submitted to a potential step and interpreted the phase gap between the original and reflected term in the tunnel effect case as a delay in the reflection of the particle. At the same time in [35], Régnier extended this technique to a two-dimensional problem which had been first studied from a spectral point of view by Croc and Dermenjian.

Let us finally quote the paper by Nicaise and Valein [32] on stabilization of the one-dimensional wave equation with a delay term in the feedbacks. They use the same method as we do in this paper (technique developed by von Below in [8]) to get the characteristic equation associated to the eigenvalues and apply this spectral analysis to stabilization.

2. Preliminaries

2.1. Terminology of networks

Let us first introduce some notations and definitions which will be used throughout the rest of the paper, in particular some which are linked to the notion of C^v -networks, $v \in \mathbb{N}$ (as introduced in [10] and recalled in [21]):

All graphs considered here are nonempty, finite and simple. Let Γ be a connected topological graph embedded in \mathbb{R}^m , $m \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, with n_0 vertices and N edges $((n_0, N) \in (\mathbb{N}^*)^2)$.

We split the set E of vertices as follows: $E = E_{\text{int}} \cup E_{\text{ext}}$ where $E_{\text{int}} = \{E_i: 1 \leq i \leq n\}$ is the set of interior vertices and $E_{\text{ext}} = \{E_i: n+1 \leq i \leq n_0\}$ the set of exterior vertices of Γ .

Let $K = \{k_j: 1 \leq j \leq N\}$ be the set of the edges of Γ . Each edge k_j is a Jordan curve in \mathbb{R}^m and is assumed to be parametrized by its arc length x_j such that the parametrization

$$\pi_j: [0, l_j] \rightarrow k_j: \quad x_j \mapsto \pi_j(x_j)$$

is v -times differentiable i.e. $\pi_j \in C^v([0, l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$. The length of the edge k_j is l_j .

The C^v -network G associated with Γ is then defined as the union

$$G = \bigcup_{j=1}^N k_j.$$

The valency of each vertex E_i is the number of edges containing the vertex E_i and is denoted by $\gamma(E_i)$. Clearly it holds $E_{\text{int}} = \{E_i: \gamma(E_i) > 1\}$ and $\partial E = E_{\text{ext}} = \{E_i \in E: \gamma(E_i) = 1\}$. For shortness, we later on denote by I_{int} (respectively I_{ext}) the set of indices corresponding

to the interior (respectively exterior) vertices i.e. $I_{\text{int}} = \{i: i \in \{1, \dots, n\}\}$ and $I_{\text{ext}} = \{i: i \in \{n+1, \dots, n_0\}\}$. For each vertex E_i , we also denote by $N_i = \{j \in \{1, \dots, N\}: E_i \in k_j\}$ the set of edges adjacent to E_i . The incidence matrix $D = (d_{ij})_{n_0 \times N}$ is defined by

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(l_j) = E_i, \\ -1 & \text{if } \pi_j(0) = E_i, \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix $\mathcal{E} = (e_{ih})_{n_0 \times n_0}$ of Γ is given by

$$e_{ih} = \begin{cases} 1 & \text{if there exists an edge } k_{s(i,h)} \text{ between } E_i \text{ and } E_h, \\ 0 & \text{otherwise.} \end{cases}$$

For a function $u: G \rightarrow \mathbb{R}$ we set $u_j = u \circ \pi_j: [0, l_j] \rightarrow \mathbb{R}$ its restriction to the edge k_j . We further use the abbreviations:

$$\begin{cases} u_j(E_i) = u_j(\pi_j^{-1}(E_i)), \\ u_{jx_j}(E_i) = \frac{du_j}{dx_j}(\pi_j^{-1}(E_i)), \\ u_{jx_j^{(n)}}(E_i) = \frac{d^n u_j}{dx_j^n}(\pi_j^{-1}(E_i)). \end{cases}$$

2.2. Data and framework

Following Castro and Zuazua [17], we study a linear system modelling the vibrations of beams connected by point masses but with N beams (instead of two) and n point masses (instead of one). To this end, let us fix a C^4 -network G such that $E_{\text{ext}} \neq \emptyset$. For each edge k_j (representing a beam of our network of beams), we fix mechanical constants: $m_j > 0$ (the mass density of the beam k_j) and $E_j I_j > 0$ (the flexural rigidity of k_j). We set $a_j = \frac{E_j I_j}{m_j}$. For each interior vertex $E_i \in E_{\text{int}}$, we fix the mass $M_i > 0$ ($1 \leq i \leq n$).

So the scalar functions $u_j(x, t)$ and $z_i(t)$ for $x \in G$ and $t > 0$ contain the information on the vertical displacements of the beams ($1 \leq j \leq N$) and of the point masses ($1 \leq i \leq n$). Our aim is to study the spectrum of the spatial operator (involved in the evolution problem) which is defined as follows.

First define the inner product $(\cdot, \cdot)_H$ on $H = \prod_{j=1}^N L^2((0, l_j)) \times \mathbb{R}^n$ by

$$((u, z), (w, s))_H = \sum_{j=1}^N \int_0^{l_j} u_j(x_j) w_j(x_j) dx_j + \sum_{i=1}^n M_i z_i s_i$$

and define the operator \mathcal{A} on the Hilbert space H endowed with the above inner product, by

$$\begin{cases} D(\mathcal{A}) = \{(u, z) \in H: u_j \in H^4((0, l_j)) \text{ satisfying (2) to (6) hereafter}\}, \\ \forall (u, z) \in D(\mathcal{A}), \quad \mathcal{A}(u, z) = \left((a_j u_{jx_j^{(4)}})_{j=1}^N, -\frac{1}{M_i} \left(\sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) \right)_{i=1}^n \right), \end{cases} \quad (1)$$

where $\frac{\partial u_j}{\partial v_j}(E_i) = d_{ij} u_{jx_j}(E_i)$ means the exterior normal derivative of u_j at E_i .

$$u_j(E_i) = z_i, \quad \forall i \in I_{\text{int}}, \quad \forall j \in N_i, \quad (2)$$

$$\sum_{j \in N_i} \frac{\partial u_j}{\partial v_j}(E_i) = 0, \quad \forall i \in I_{\text{int}}, \quad (3)$$

$$a_l \frac{\partial^2 u_l}{\partial v_l^2}(E_i) = a_j \frac{\partial^2 u_j}{\partial v_j^2}(E_i), \quad \forall i \in I_{\text{int}}, (l, j) \in N_i^2, \quad (4)$$

$$u_j(E_i) = 0, \quad \forall i \in I_{\text{ext}}, \forall j \in N_i, \quad (5)$$

$$\frac{\partial^2 u_j}{\partial v_j^2}(E_i) = 0, \quad \forall i \in I_{\text{ext}}, \forall j \in N_i. \quad (6)$$

Notice that the conditions (2) imply the continuity of u on G . The conditions (3) and (4) are transmission conditions at the interior nodes and (5) and (6) are boundary conditions.

Lemma 1 (Properties of the operator \mathcal{A}). *The operator \mathcal{A} defined by (1) is a nonnegative self-adjoint operator with a compact resolvent.*

Proof. The reason for \mathcal{A} to be a self-adjoint operator with a compact resolvent, is that it is the Friedrichs extension of the triple (H, V, a) defined by

$$V = \left\{ U = (u, z) \in \prod_{j=1}^N H^2((0, l_j)) \times \mathbb{R}^n : \text{satisfying (2), (3), (5)} \right\}$$

which is a Hilbert space endowed with the inner product

$$(U, W)_V = ((u, z), (w, s))_V = \sum_{j=1}^N (u_j, w_j)_{H^2((0, l_j))} + \sum_{i=1}^n M_i z_i s_i,$$

where $(\cdot, \cdot)_{H^2((0, l_j))}$ is the usual inner product on $(0, l_j)$ and

$$a(U, W) = \sum_{j=1}^N a_j \int_0^{l_j} u_{jx_j^{(2)}}(x_j) w_{jx_j^{(2)}}(x_j) dx_j. \quad (7)$$

Let us prove this result. The assumptions of Friedrichs Theorem are clearly satisfied (cf. Theorem 2.2.1 of [1]): a is a continuous, symmetric, coercitive sesquilinear form and V and H are Hilbert spaces such that V is densely embedded into H . Thus the operator $(\mathcal{A}^V, D(\mathcal{A}^V))$ is self-adjoint where $D(\mathcal{A}^V)$ is defined by

$$D(\mathcal{A}^V) = \{U \in V : \exists f \in H, a(U, W) = (f, W)_H, \forall W \in V\}.$$

Two parts integrations in the expression of the sesquilinear form a lead to $U \in D(\mathcal{A}^V) \Leftrightarrow U \in V$ and $\exists f = (g_1, \dots, g_N, h_1, \dots, h_n) \in H$ such that for any $W = (w, s) \in V$:

$$\begin{aligned} & \sum_{j=1}^N a_j \int_0^{l_j} u_j^{(4)}(x_j) w_j(x_j) dx_j + \sum_{j=1}^N a_j \left[\frac{\partial^2 u_j}{\partial v_j^2}(x_j) w'_j(x_j) \right]_{x_j \in \partial k_j} \\ & - \sum_{j=1}^N a_j \left[\frac{\partial^3 u_j}{\partial v_j^3}(x_j) w_j(x_j) \right]_{\partial k_j} \end{aligned}$$

$$= \sum_{j=1}^N \int_0^{l_j} g_j(x_j) w_j(x_j) dx_j + \sum_{i=1}^n M_i h_i s_i.$$

Now, W belongs to V so it satisfies (2), (3) and (5).

Due to (2) and (5), $U \in D(\mathcal{A}^V) \Leftrightarrow U \in V$ and $\exists f = (g, h) \in H$ such that for any $W = (w, s) \in V$:

$$\begin{aligned} & \sum_{j=1}^N a_j \int_0^{l_j} u_j^{(4)}(x_j) w_j(x_j) dx_j + \sum_{j=1}^N a_j \left[\frac{\partial^2 u_j}{\partial v_j^2}(x_j) w_j'(x_j) \right]_{\partial k_j} - \sum_{i=1}^n \sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) s_i \\ &= \sum_{j=1}^N \int_0^{l_j} g_j(x_j) w_j(x_j) dx_j + \sum_{i=1}^n M_i h_i s_i. \end{aligned}$$

Then the condition (3) on the interior nodes implies U satisfies (4) and the absence of condition on w_j' at the exterior nodes implies U satisfies (6) (those conditions classically follow from an appropriate choice of W). Thus $U \in D(\mathcal{A}^V) \Leftrightarrow U \in V$ and satisfies (4) and (6) and $\exists f = (g, h) \in H$ such that for any $W = (w, s) \in V$:

$$\begin{aligned} & \sum_{j=1}^N a_j \int_0^{l_j} u_j^{(4)}(x_j) w_j(x_j) dx_j - \sum_{i=1}^n \sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) s_i \\ &= \sum_{j=1}^N \int_0^{l_j} g_j(x_j) w_j(x_j) dx_j + \sum_{i=1}^n M_i h_i s_i. \end{aligned}$$

Hence the expression for the operator \mathcal{A}^V which coincides with that of \mathcal{A} . Both domains also coincide.

There remains to prove the positiveness of \mathcal{A} . It follows from the equivalence between $a(u, u)$ and $(u, u)_V$. This is due to the fact that G has at least one exterior vertex as the next lemma shows:

Lemma 2. *Let V , $(\cdot, \cdot)_V$ and a be defined as above. There exists $C > 0$ such that*

$$(U, U)_V \leq C \cdot a(U, U), \quad \forall U \in V. \quad (8)$$

Proof. Using a standard contradiction argument with the help of the compact embedding of V into H (the embedding of $H^2(\Omega)$ into $L^2(\Omega)$ is compact for a bounded Ω , due to Rellich's Theorem), (8) holds if both conditions $U \in V$ and $a(U, U) = 0$ imply $U = 0$. Now such a $U = (u, z)$ satisfies $z = 0$ and is a polynomial of order 1 on each edge. From the interior condition (2) and the Dirichlet conditions (5), we get $u = 0$. \square

Thus Lemma 1 is proved. \square

3. Spectrum

Our aim is to characterize the spectrum $\sigma(\mathcal{A})$ of \mathcal{A} . According to Lemma 2 this spectrum is positive and discrete. As in [10] (see also [22]), we shall rewrite the eigenvalue problem into an equivalent matrix differential value problem.

3.1. Characterization of the eigenelements

Let $\lambda^2 \in \sigma(\mathcal{A})$ ($\lambda > 0$) be an eigenvalue of \mathcal{A} with associated eigenvector $U = (u, z) \in D(\mathcal{A})$. Then u satisfies the transmission and boundary conditions (2)–(6) of Section 2.2 and

$$\begin{cases} a_j u_{j x_j^{(4)}} = \lambda^2 u_j & \text{on } (0, l_j), \forall j \in \{1, \dots, N\}, \\ \sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) = \lambda^2 M_i z_i, & \forall i \in I_{\text{int}}, \\ u_j \in H^4((0, l_j)), & \forall j \in \{1, \dots, n\}. \end{cases}$$

Notation.

1. We shall use the Hadamard product of matrices defined by $X \cdot Y = (x_{ih} y_{ih})_{n_0 \times n_0}$ where the matrices X and Y are $X = (x_{ih})_{n_0 \times n_0}$ and $Y = (y_{ih})_{n_0 \times n_0}$.
2. For any function $p: \mathbb{R} \rightarrow \mathbb{R}$, we define $p(X) = (p_{ih})_{n_0 \times n_0}$ where p_{ih} is defined by

$$p_{ih} = \begin{cases} p(x_{ih}) & \text{if } e_{ih} = 1, \\ 0 & \text{if } e_{ih} = 0. \end{cases}$$

In particular if $p(x) = x^r$ then we write $X^{(r)}$ instead of $p(X)$.

3. We set $e = (1)_{n_0 \times 1}$, $e_{\text{int}} = (\varepsilon_i)_{n_0 \times 1}$ (respectively e_{ext}) with $\varepsilon_i = 1$ if $i \in I_{\text{int}}$ (respectively $i \in I_{\text{ext}}$), else $\varepsilon_j = 0$. For any vector v of \mathbb{R}^{n_0} we define the diagonal matrix $\text{Diag}(v) = (\delta_{ih} v_i)_{n_0 \times n_0}$ and the vectors

$$\begin{cases} v^{\text{int}} = v \cdot e_{\text{int}}, \\ v^{\text{ext}} = v \cdot e_{\text{ext}}. \end{cases}$$

4. We finally introduce the matrices $A = (a_{s(i,h)} e_{ih})_{n_0 \times n_0}$, $L = (l_{s(i,h)} e_{ih})_{n_0 \times n_0}$ and $B = L \cdot A^{(-1/4)}$ where $k_{s(i,h)}$ is the edge joining the vertex E_i to the vertex E_h and the point mass-matrix $M = (M_i)_{n \times 1}$ with $M_i = 0$ when $i \in I_{\text{ext}}$.

Example 3. Consider the graph (represented in Fig. 1) with $N = 3$ edges and $n_0 = 4$ vertices with two interior vertices E_1 and E_2 (so $n = 2$ and $I_{\text{int}} = \{1; 2\}$) and two exterior vertices E_3 and E_4 ($I_{\text{ext}} = \{3; 4\}$). The edge k_1 links E_3 to E_1 , k_2 links E_1 to E_2 and k_3 links E_2 to E_4 . The incidence matrix D has four lines and three columns, the matrices \mathcal{E} and B are square with order 4 and symmetric and M has four lines and one column:

$$D = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_2 & b_1 & 0 \\ b_2 & 0 & 0 & b_3 \\ b_1 & 0 & 0 & 0 \\ 0 & b_3 & 0 & 0 \end{pmatrix}$$

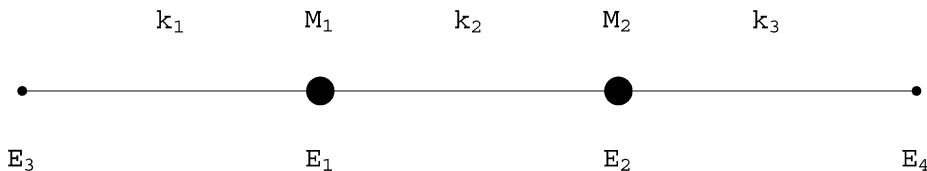


Fig. 1. Graph with $N = 3$ edges.

and

$$M = \begin{pmatrix} M_1 \\ M_2 \\ 0 \\ 0 \end{pmatrix},$$

where $b_j = l_j \cdot a_j^{(-1/4)}$ for any $j \in \{1; 2; 3\}$.

$$e = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad e_{\text{int}} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_{\text{ext}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

and if

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \varphi^3 \\ \varphi^4 \end{pmatrix}, \quad \varphi^{\text{int}} = \begin{pmatrix} \varphi^1 \\ \varphi^2 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi^{\text{ext}} = \begin{pmatrix} 0 \\ 0 \\ \varphi^3 \\ \varphi^4 \end{pmatrix}.$$

Definition 4. To any function u defined on the graph G is associated the matrix function

$$U : [0, 1] \rightarrow \mathbb{R}^{n_0 \times n_0}, \quad x \mapsto U(x) = (u_{ih}(x))_{n_0 \times n_0},$$

where

$$u_{ih}(x) = e_{ih} u_{s(i,h)} \left[l_{s(ih)} \left(\frac{1 + d_{s(i,h)}}{2} - d_{s(i,h)} x \right) \right],$$

i.e.

$$u_{ih}(x) = \begin{cases} e_{ih} u_{s(i,h)} l_{s(ih)} \cdot (1 - x) & \text{if } d = 1, \\ e_{ih} u_{s(i,h)} l_{s(ih)} \cdot x & \text{if } d = -1, \\ e_{ih} u_{s(i,h)} l_{s(ih)} \cdot \frac{1}{2} & \text{if } d = 0. \end{cases}$$

Lemma 5 (Characterization of the eigenelements). $(u, z) \in D(\mathcal{A})$ is a eigenvector of \mathcal{A} associated to the eigenvalue λ^2 ($\lambda > 0$), if and only if U (defined above) is a solution of the differential problem (9) to (15):

$$u_{ih} \in H^4((0, 1)) \quad \text{and} \quad (e_{ih} = 0 \Rightarrow u_{ih} = 0), \quad \forall (i, h) \in \{1, \dots, n\}^2, \quad (9)$$

$$L^{(-4)} \cdot A \cdot U^{(4)}(x) = \lambda^2 U(x), \quad \forall x \in (0, 1), \quad (10)$$

$$\exists \varphi_0 \in \mathbb{R}^n: \quad U(0) = (\varphi_0 e^T) \cdot \mathcal{E}, \quad \varphi_0^{\text{ext}} = 0, \quad (11)$$

$$\exists \varphi_2 \in \mathbb{R}^n: \quad U''(0) = L^{(2)} \cdot A^{(-1)} \cdot (\varphi_2 e^T) \cdot \mathcal{E}, \quad \varphi_2^{\text{ext}} = 0, \quad (12)$$

$$([L^{-1} \cdot A \cdot U'(0)]e) \cdot e_{\text{int}} = 0, \quad (13)$$

$$([L^{-3} \cdot A \cdot U'''(0)]e) \cdot e_{\text{int}} = \lambda^2 M \cdot \varphi_0, \quad (14)$$

$$U^T(1 - x) = U(x). \quad (15)$$

Proof. The result comes from a simple rewriting of problem (1) to (6) of Section 2.2, using the above definition of U .

(10) and (14) express the operator: (10) corresponds to the vibrations of the beams and (14) to those of the point masses (cf. (1)).

(11), (12) and (13) translate the transmission conditions (2), (4) and (3), respectively.

The conditions $\varphi_0^{\text{ext}} = 0$ and $\varphi_2^{\text{ext}} = 0$ of Eqs. (11) and (12) are the boundary conditions (5) and (6).

To finish with, (15) is a property of “symmetry” of the matrix U which clearly follows from its definition. \square

3.2. The characteristic equation

Following the method developed in [10] and [22], we shall show that the differential problem (9)–(15) can be reduced to an algebraic system whose nontrivial solutions determine nontrivial eigenvectors.

Lemma 6 (System of fundamental solutions of the differential equation (10)). *Let the four functions $e_i : [0, 1] \rightarrow \mathbb{R}^{n_0 \times n_0}$ be defined, for $0 \leq i \leq 3$, by*

$$\begin{cases} e_0^\lambda(x) = \frac{1}{2} \{ \cos(\sqrt{\lambda} Bx) + \cosh(\sqrt{\lambda} Bx) \}, \\ e_1^\lambda(x) = \frac{1}{2\sqrt{\lambda}} B^{(-1)} \cdot \{ \sin(\sqrt{\lambda} Bx) + \sinh(\sqrt{\lambda} Bx) \}, \\ e_2^\lambda(x) = \frac{1}{2\lambda} B^{(-2)} \cdot \{ -\cos(\sqrt{\lambda} Bx) + \cosh(\sqrt{\lambda} Bx) \}, \\ e_3^\lambda(x) = \frac{1}{2\lambda^{\frac{3}{2}}} B^{(-3)} \cdot \{ -\sin(\sqrt{\lambda} Bx) + \sinh(\sqrt{\lambda} Bx) \}, \end{cases}$$

where B is the symmetric matrix $B = L \cdot A^{(-1/4)}$. They form a system of fundamental solutions of the differential equation (10) satisfying

$$e_i^{(j)}(0) = \delta_{ij} \mathcal{E}, \quad \forall (i, j) \in \{0, \dots, 3\}^2.$$

Consequently, if (u, z) is an eigenvector of \mathcal{A} with eigenvalue $\lambda^2 > 0$ then U admits the expansion

$$U(x) = \sum_{i=0}^3 \Phi_i \cdot e_i^\lambda(x) \tag{16}$$

with $\Phi_i \in \mathbb{R}^{n_0 \times n_0}$.

Proof. Simple calculations analogous to those in [22]. \square

3.2.1. Spectrum for some particular cases

Theorem 7 (Characteristic equation for a never vanishing $\sin(l_j a_j^{-1/4} \sqrt{\lambda})$). *Let $\lambda^2 > 0$ be an eigenvalue of \mathcal{A} . If the following statement holds:*

$$\sin(l_j a_j^{-1/4} \sqrt{\lambda}) \neq 0, \quad \forall j \in \{1, \dots, N\},$$

then λ satisfies the characteristic equation

$$\det \mathcal{D}(\sqrt{\lambda}, \mathcal{A}, L, M, \mathcal{E}) = 0, \tag{17}$$

where the $2n \times 2n$ matrix $\mathcal{D}(\sqrt{\lambda}, \mathcal{A}, L, M, \mathcal{E})$ is defined by

$$\mathcal{D}(\sqrt{\lambda}, \mathcal{A}, L, M, \mathcal{E}) = \begin{pmatrix} \mathcal{D}_{11}^{\text{int}} & \mathcal{D}_{12}^{\text{int}} \\ \mathcal{D}_{21}^{\text{int}} - 2\sqrt{\lambda} \text{Diag}[M] & \mathcal{D}_{11}^{\text{int}} \end{pmatrix}.$$

$\mathcal{D}_{ij}^{\text{int}}$ being the restriction to the first $n \times n$ lines and columns of $\mathcal{D}_{ij} \in \mathbb{R}^{n_0 \times n_0}$ given by

$$\begin{cases} \mathcal{D}_{11} = \text{Diag}([-C^{(-1)} \cdot A^{(-\frac{1}{4})} \cdot (\sin(\sqrt{\lambda}B) \cdot \cosh(\sqrt{\lambda}B) + \cos(\sqrt{\lambda}B) \cdot \sinh(\sqrt{\lambda}B))]e \\ \quad + C^{(-1)} \cdot A^{(-\frac{1}{4})} \cdot (\sin(\sqrt{\lambda}B) + \sinh(\sqrt{\lambda}B)), \\ \mathcal{D}_{12} = \text{Diag}([C^{(-1)} \cdot A^{(-\frac{3}{4})} \cdot (-\sin(\sqrt{\lambda}B) \cosh(\sqrt{\lambda}B) + \cos(\sqrt{\lambda}B) \cdot \sinh(\sqrt{\lambda}B))]e \\ \quad + C^{(-1)} \cdot A^{(-\frac{3}{4})} \cdot (\sin(\sqrt{\lambda}B) - \sinh(\sqrt{\lambda}B)), \\ \mathcal{D}_{21} = \text{Diag}([C^{(-1)} \cdot A^{(\frac{1}{4})} \cdot (-\sin(\sqrt{\lambda}B) \cdot \cosh(\sqrt{\lambda}B) + \cos(\sqrt{\lambda}B) \cdot \sinh(\sqrt{\lambda}B))]e \\ \quad + C^{(-1)} \cdot A^{(\frac{1}{4})} \cdot (\sin(\sqrt{\lambda}B) - \sinh(\sqrt{\lambda}B)), \end{cases}$$

where $C = \sin(\sqrt{\lambda}B) \cdot \sinh(\sqrt{\lambda}B)$ and B is the symmetric matrix $B = L \cdot A^{(-1/4)}$.

Moreover, the associated eigenvector (u, z) is such that the matrix function U has the expansion $U(x) = \sum_{i=0}^3 \Phi_i \cdot e_i^\lambda(x)$ with

$$\Phi_0 = (\varphi_0 e^T) \cdot \mathcal{E}, \quad \Phi_2 = (\varphi_2 e^T) \cdot \mathcal{E}, \quad \varphi_0^{\text{ext}} = \varphi_2^{\text{ext}} = 0$$

and

$$\begin{pmatrix} \lambda \varphi_0 \\ \varphi_2 \end{pmatrix} \in \text{Ker } \mathcal{D}(\sqrt{\lambda}, A, L, M, \mathcal{E}). \quad (18)$$

As for Φ_1 and Φ_3 , they are uniquely determined as the solution of a system in Φ_0 and Φ_2 (see (21) in the proof).

Proof. Using Lemma 2.5 of [22], it follows that U satisfies (15) if and only if U satisfies

$$\begin{cases} U(1) = U(0)^T, \\ U''(1) = U''(0)^T, \end{cases}$$

and due to the way the $e_i^{(j)}$'s have been constructed (cf. Lemma 6), it holds:

$$U^{(j)}(0) = \Phi_j \cdot \mathcal{E}, \quad \forall j \in \{0, \dots, 3\}. \quad (19)$$

Thus the above system is equivalent to

$$\begin{cases} \Phi_1 \cdot f_1 + \Phi_3 \cdot f_3 = \Phi_0^T - \Phi_0 \cdot f_0 - \Phi_2 \cdot f_2, \\ \Phi_1 \cdot \lambda^2 B^{(4)} \cdot f_3 + \Phi_3 \cdot f_1 = \Phi_2^T - \Phi_0 \cdot \lambda^2 B^{(4)} \cdot f_2 - \Phi_2 \cdot f_0 \end{cases} \quad (20)$$

with $f_i = e_i^\lambda(1)$, for $i \in \{0, \dots, 3\}$.

Since $\sin(l_j a_j^{-1/4} \sqrt{\lambda}) \neq 0$, $\forall j \in \{1, \dots, N\}$, the matrix $C = \sin(\sqrt{\lambda}B) \cdot \sinh(\sqrt{\lambda}B)$ is invertible in the Hadamard sense. Now $f_1^2 - f_3 \cdot \lambda^2 B^{(4)} \cdot f_3 = C \lambda^{-1} B^{(-2)}$. Thus the system (20) is equivalent to the following one:

$$\begin{cases} C \cdot \Phi_1 = \lambda B^{(2)} \cdot [f_1 \cdot \Phi_0^T + (\lambda^2 \cdot B^{(4)} \cdot f_3 \cdot f_2 - f_1 \cdot f_0) \cdot \Phi_0 \\ \quad + (f_3 \cdot f_0 - f_1 \cdot f_2) \Phi_2 - f_3 \cdot \Phi_2^T], \\ C \cdot \Phi_3 = \lambda B^{(2)} \cdot [-\lambda^2 B^{(4)} \cdot f_3 \cdot \Phi_0^T + f_1 \cdot \Phi_2^T + B^{(4)} \cdot (f_3 \cdot f_0 - f_1 \cdot f_2) \Phi_0 \\ \quad + (\lambda^2 \cdot B^{(4)} \cdot f_3 \cdot f_2 - f_1 \cdot f_0) \cdot \Phi_2]. \end{cases} \quad (21)$$

Consequently Φ_1 and Φ_3 are uniquely determined by this system in Φ_0 and Φ_2 . There remains to express the conditions (13) and (14).

Using (21) in (13) and (14) with the conditions (11) and (12) and with the help of the easily checked identities

$$(B \cdot \mathcal{E} \cdot (e\varphi^T))e = B\varphi \quad \text{and} \quad (B \cdot \mathcal{E} \cdot (\varphi e^T))e = [\text{Diag}(Be)]\varphi,$$

we get

$$\begin{pmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{D}_{21} - 2\sqrt{\lambda} \text{Diag}[M] & \mathcal{D}_{11} \end{pmatrix} \begin{pmatrix} \lambda\varphi_0 \\ \varphi_2 \end{pmatrix} = 0$$

which is equivalent to (18) as soon as the boundary conditions $\varphi_0^{\text{ext}} = \varphi_2^{\text{ext}} = 0$ are satisfied. \square

Remark 8.

- Note that $\Phi_0 = (\varphi_0 e^T) \cdot \mathcal{E}$ with $\varphi_0^{\text{ext}} = 0$ means, on our previous example of Section 3.1, that

$$\varphi_0 = \begin{pmatrix} \varphi_0^1 \\ \varphi_0^2 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_0 = \begin{pmatrix} 0 & \varphi_0^1 & \varphi_0^1 & 0 \\ \varphi_0^2 & 0 & 0 & \varphi_0^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- The characteristic equation looks very much like that of [22]. The additional term $(-2\sqrt{\lambda} \text{Diag}[M])$ comes from the point masses we have added here.
- Notice that, if U is known, then u is determined as well as z , which is obtained through

$$\sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v^3}(E_i) = \lambda^2 z_i, \quad \forall i \in I_{\text{int}}.$$

The possibility for $\sin(l_j a_j^{-1/4} \sqrt{\lambda})$ to vanish for some values of j has been excluded in the above theorem. Since it becomes hard to deal with the general situation, let us first envisage a special case of vanishing for $\sin(l_j a_j^{-1/4} \sqrt{\lambda})$, which is the case of a chain of N identical beams. The characteristic equation is computable as well as the eigenvectors:

Theorem 9 (Characteristic equation for a possibly vanishing $\sin(l_j a_j^{-1/4} \sqrt{\lambda})$ and a graph with N identical branches). *The mechanical constants are assumed to be identical for all the beams i.e. $L = l \cdot \mathcal{E}$, $A = a \cdot \mathcal{E}$, so $B = b \cdot \mathcal{E}$ with $l > 0$, $a > 0$ and $b = l \cdot a^{-1/4} > 0$.*

Let $\lambda^2 > 0$ be an eigenvalue of \mathcal{A} . If $\sin(\sqrt{\lambda} B) = 0$, then λ satisfies the characteristic equation

$$\det \mathcal{D}'(\lambda, \mathcal{A}, L, M, \mathcal{E}) = 0, \tag{22}$$

where the $n_0 \times n_0$ matrix $\mathcal{D}'(\lambda, \mathcal{A}, L, M, \mathcal{E})$ is the restriction to the first $n_0 \times n_0$ lines and columns of the $n \times n$ matrix

$$f_3^{(-1)} - \text{Diag}[(f_3^{-1} \cdot f_0)e] - \text{Diag}[\lambda b^2 (f_3^{-1} \cdot f_2)e] - \lambda^2 \text{Diag}[M]$$

with

$$\begin{cases} f_0 = \frac{1}{2} [\cosh(\sqrt{\lambda}b) - \alpha] \cdot \mathcal{E}, \\ f_2 = \frac{1}{2\lambda b^2} [\cosh(\sqrt{\lambda}b) + \alpha] \cdot \mathcal{E}, \\ f_3^{(-1)} = \frac{2\lambda^{3/2}b^3}{\sinh(\sqrt{\lambda}b)} \cdot \mathcal{E} \end{cases}$$

and $\alpha = -\cos(\sqrt{\lambda}B)$.

Moreover, the associated eigenvector (u, z) is such that the matrix function U has the expansion $U(x) = \sum_{i=0}^3 \Phi_i \cdot e_i^\lambda(x)$ with

$$\Phi_0 = (\varphi_0 e^T) \cdot \mathcal{E}, \quad \Phi_2 = (\varphi_2 e^T) \cdot \mathcal{E}, \quad \varphi_0^{\text{ext}} = \varphi_2^{\text{ext}} = 0$$

and

$$\begin{cases} \varphi_0^{\text{int}} \in \text{Ker } \mathcal{D}'(\lambda, A, L, M, \mathcal{E}), \\ \varphi_2 = \lambda b^2 \varphi_0. \end{cases} \quad (23)$$

As for Φ_1 and Φ_3 , they are given by (24) in the proof where the expression of X is calculated using the system of equations following from (25) and (31).

Furthermore the dimension of the eigenspace associated to the eigenvalue λ^2 is

$$\dim(\text{Ker } \mathcal{D}'(\lambda, A, L, M, \mathcal{E})) + 1.$$

Proof. In this theorem the case of a possibly vanishing $\sin(l_j a_j^{-1/4} \sqrt{\lambda})$ is envisaged. But since the beams are chosen to be identical and since $b = l \cdot a^{-1/4} > 0$, $\sin(l_j a_j^{-1/4} \sqrt{\lambda}) = \sin(l \cdot a^{-1/4} \sqrt{\lambda}) = \sin(\sqrt{\lambda}b)$. So the assumption is translated into $\sin(\sqrt{\lambda}B) = 0$.

The system (20) of the above proof of Theorem 7 still holds but, since $C = \sin(\sqrt{\lambda}B) \cdot \sinh(\sqrt{\lambda}B)$ is not invertible in the Hadamard sense ($C = 0$), Φ_1 and Φ_3 are not uniquely determined anymore. In fact, (20) is equivalent to the existence of $X \in \mathbb{R}^{n \times n}$ such that

$$\begin{cases} \Phi_1 = X \cdot f_3, \\ \Phi_3 = f_3^{(-1)} \cdot (\Phi_0^T - \Phi_0 \cdot f_0 - \Phi_2 \cdot f_2) - X \cdot f_1. \end{cases} \quad (24)$$

Lemma 10 (“Symmetry” property of X). Let α and X be the $n \times n$ matrices defined by $\alpha = -\cos(\sqrt{\lambda}B)$ and X is any solution of (24). Suppose that $\sin(\sqrt{\lambda}B) = 0$. Then $\alpha = \pm 1$ and

$$-X + \alpha X^T = \frac{2\lambda^2 b^4}{\sinh(\sqrt{\lambda}b)} (\alpha + \cosh(\sqrt{\lambda}B)) \cdot [\Phi_0^T - \alpha \lambda^{-1} b^{-2} \Phi_2]. \quad (25)$$

Proof. The first part of the lemma is clear. Now that $\sin(\sqrt{\lambda}B) = 0$, Lemma 2.5 of [22] implies that U satisfies (15) if and only if U satisfies:

$$\begin{cases} U(1) = U(0)^T, \\ U'(1) = -U'(0)^T. \end{cases} \quad (26)$$

Then the right-hand side of the second equation of (26) is $-\Phi_1^T = -X^T \cdot f_3$, due to (24) and to the symmetry of f_3 (which follows from that of B). The left-hand side is the sum $\sum_{i=0}^3 \varphi_i \cdot (e_i^\lambda)'(1)$. After some calculation, using $f_i = e_i^\lambda(1)$ where the e_i^λ 's are given by Lemma 6, the second equation of (26) is equivalent to

$$-X^T \cdot f_3 + X \cdot (f_1 \cdot f_2 - f_0 \cdot f_3) = \lambda^2 B^{(4)} \cdot f_3 \cdot \Phi_0 + f_1 \cdot \Phi_2 + f_2 \cdot f_3^{(-1)} \cdot (\Phi_0 - f_0 \cdot \Phi_0 - f_2 \cdot \Phi_2). \quad (27)$$

Now since $\sin(\sqrt{\lambda}B) = 0$, $f_3 = e_3^\lambda(1) = \frac{1}{2\lambda^{3/2}}B^{(-3)} \cdot \sinh(\sqrt{\lambda}B)$ and it follows that $f_1 \cdot f_2 - f_0 \cdot f_3 = f_3 \cdot e$. Easy computations for the right-hand side of (27) lead to

$$\begin{aligned} & (-X^T + \alpha X) \left(\frac{1}{2\lambda^{3/2}} B^{(-3)} \cdot \sinh(\sqrt{\lambda}B) \right) \\ &= \frac{\sqrt{\lambda}b}{\sinh(\sqrt{\lambda}b)} (\alpha + \cosh(\sqrt{\lambda}B)) \cdot [\Phi_0^T - \alpha\lambda^{-1}b^{-2}\Phi_2]. \end{aligned} \quad (28)$$

At last the vanishing of $\sin(\sqrt{\lambda}B)$ implies that both equations of the system (20) are equivalent to one another i.e. both right-hand sides are proportional. In fact:

$$\Phi_0 - \cos(\sqrt{\lambda}B) \cdot \Phi_0^T = \lambda^{-1}b^{-2}(\Phi_2 - \cos(\sqrt{\lambda}B) \cdot \Phi_2^T), \quad (29)$$

which can also be rewritten as

$$\Phi_0 + \alpha \cdot \Phi_0^T = \lambda^{-1}b^{-2}(\Phi_2 + \alpha \cdot \Phi_2^T). \quad (30)$$

Thus $[\Phi_0^T - \alpha\lambda^{-1}b^{-2}\Phi_2]^T = -\alpha[\Phi_0^T - \alpha\lambda^{-1}b^{-2}\Phi_2]$. And (25) follows from all that. \square

Let us now come back to the proof of the theorem. As in the proof of Theorem 7, we use (24) in the transmission conditions (13) and (14) without forgetting (11) and (12) and in particular the boundary conditions contained in them.

- Condition (13) is $([L^{-1} \cdot A \cdot U'(0)]e) \cdot e_{\text{int}} = 0$. Using (19) with $j = 1$ and the expression of Φ_1 given by (24) as $X \cdot f_3$ that is to say

$$\Phi_1 = \left(\frac{\sinh(\sqrt{\lambda}b)}{2\lambda^{3/2}} \right) X \cdot \mathcal{E},$$

(13) is equivalent to

$$\left(\frac{l^{-1}a \sinh(\sqrt{\lambda}b)}{2\lambda^{3/2}} \right) (Xe) \cdot e_{\text{int}} = 0$$

and, since $\frac{l^{-1}a \sinh(\sqrt{\lambda}b)}{2\lambda^{3/2}} \neq 0$, (13) is equivalent to

$$(Xe) \cdot e_{\text{int}} = 0. \quad (31)$$

- Condition (14) is $([L^{-3} \cdot A \cdot U'''(0)]e) \cdot e_{\text{int}} = \lambda^2 M \cdot \Phi_0$. Using (19) with $j = 3$ and the expression of Φ_3 given by (24), (14) is equivalent to

$$((f_1 \cdot X \cdot \mathcal{E})e) \cdot e_{\text{int}} = ((f_3^{(-1)} \cdot (\Phi_0^T - \Phi_0 \cdot f_0 - \Phi_2 \cdot f_2))e) \cdot e_{\text{int}} - \lambda^2 M \cdot \Phi_0.$$

Now $((f_1 \cdot X \cdot \mathcal{E})e) \cdot e_{\text{int}} = f_1 \cdot [(Xe) \cdot e_{\text{int}}] = 0$, due to (31). Thus (14) is equivalent to

$$((f_3^{(-1)} \cdot (\Phi_0^T - \Phi_0 \cdot f_0 - \Phi_2 \cdot f_2))e) \cdot e_{\text{int}} - \lambda^2 M \cdot \Phi_0 = 0. \quad (32)$$

To get the characteristic equation, we need another linear relationship between φ_0 and φ_2 to combine it with (32). It will come from (30) which is a consequence of the vanishing of $\sin(\sqrt{\lambda}B)$ and the boundary conditions $\varphi_0^{\text{ext}} = \varphi_2^{\text{ext}} = 0$ contained in (11) and (12). Both these conditions mean in our situation that $\varphi_0^N = \varphi_0^{N+1} = \varphi_2^N = \varphi_2^{N+1} = 0$. Writing Φ_0 as $(\varphi_0 e^T) \cdot \mathcal{E}$ and Φ_2 on the same model, we identify all the terms of the matrix involved in the left-hand side of (30) with those of the right-hand side to get:

$$\varphi_2 = \lambda b^2 \varphi_0. \quad (33)$$

Then, for the same reasons as in the proof of Theorem 7, (32) and (33) are equivalent to

$$\begin{cases} \varphi_2 = \lambda b^2 \varphi_0, \\ ((f_3^{(-1)} - \text{Diag}[(f_3^{-1} \cdot f_0)e] - \text{Diag}[\lambda b^2(f_3^{-1} \cdot f_2)e] - \lambda^2 \text{Diag}[M])\varphi_0) \cdot e_{\text{int}} = 0. \end{cases}$$

Hence the characteristic equation of the theorem. There remains to find the expressions for Φ_1 and Φ_3 i.e. to find the matrix X (knowing φ_0 and φ_2) since Φ_1 and Φ_3 can then be computed due to (24).

Let us define the support of a $m \times m$ matrix Y by $\text{supp}(Y) = \{(i, h) \in \{1, \dots, m\}^2: y_{ih} \neq 0\}$. Then the support of B and that of the f_i 's are subsets of the support of \mathcal{E} and the matrix X can be assumed to satisfy this property as well. (The values of Φ_1 and Φ_3 do not change doing so, since they are obtained through Hadamard products with the f_i 's.)

At last X satisfies (25) and (31). \square

Example 11 (*A chain of N identical branches*). Suppose that G is the graph with N edges and $(N + 1)$ vertices given by the following adjacency matrix:

$$\mathcal{E} = (e_{ih})_{(N+1) \times (N+1)} = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 & 0 & 1 \\ 1 & 0 & 1 & \ddots & \cdots & \vdots & \vdots & 0 \\ 0 & 1 & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & 0 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 1 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & 1 & \vdots \\ 0 & \vdots & & & \ddots & 1 & 0 & \vdots \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Then Theorem 9 holds and X is the $(N + 1) \times (N + 1)$ matrix:

$$X = (x_{ih})_{(N+1) \times (N+1)} = \begin{pmatrix} 0 & x_1 & 0 & \cdots & \cdots & 0 & 0 & x_{2N} \\ x_N & 0 & x_2 & \ddots & \cdots & \vdots & \vdots & 0 \\ 0 & x_{N+1} & 0 & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & x_{N+2} & \ddots & \ddots & 0 & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & x_{N-2} & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 & x_{N-1} & \vdots \\ 0 & \vdots & & & \ddots & x_{2N-2} & 0 & \vdots \\ x_{2N-1} & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}$$

with x_1 any real number and (x_2, \dots, x_{2N}) given by

$$\begin{cases} x_1 + x_{2N} = 0, \\ x_N + x_2 = 0, \\ x_{N+1} + x_3 = 0, \\ \vdots \\ x_{2N-3} + x_{N-1} = 0 \end{cases} \quad \text{and} \quad \begin{cases} -x_1 + \alpha x_N = \beta_{1,2}, \\ -x_2 + \alpha x_{N+1} = \beta_{2,3}, \\ \vdots \\ -x_{N+1} + \alpha x_{2N-2} = \beta_{N-1,N}, \\ -x_{2N} + \alpha x_{2N-1} = \beta_{1,N+1}. \end{cases}$$

The $\beta_{i,h}$'s are the terms of the matrix involved in the right-hand side of the “symmetry” property of X denoted by (25) in Lemma 10.

The $2N - 1$ equations come from the fact that X satisfies (25) and (31) which give respectively $N - 1$ and N equations. So only one term is free (x_1). Thus the dimension of the eigenspace associated to the eigenvalue λ^2 is

$$\dim(\text{Ker } \mathcal{D}'(\lambda, A, L, M, \mathcal{E})) + 1.$$

Example 12 (*A chain of $N = 3$ identical branches*). We apply Theorems 7 and 9 to the case of a chain of $N = 3$ identical branches. We set here $b_j = l_j \cdot a_j^{-1/4} = 1$ for $j = 1, 2, 3$ with the notation introduced at the beginning of Section 2.2. From Theorem 7 we get after some computation that if $\sqrt{\lambda} \neq k\pi$ then

$$\det \mathcal{D}(\sqrt{\lambda}, A, L, M, \mathcal{E}) = \frac{M_1 M_2 \lambda e^{2\sqrt{\lambda}} (3 - 4 \sin(2\sqrt{\lambda})) + o(\lambda e^{2\sqrt{\lambda}})}{(\sin(2\sqrt{\lambda}))^2 (\sinh(\sqrt{\lambda}))^2}.$$

We deduce that asymptotically the eigenvalues are of the form

$$\left(\frac{\arcsin(\frac{3}{4}) + k\pi}{2} \right)^2, \quad k \in \mathbb{N},$$

and are simple.

On the other hand, from Theorem 9, $\lambda = (k\pi)^2$ is an eigenvalue for any integer $k \in \mathbb{N} - \{0\}$. Moreover a computation on a formal calculation software gives

$$\det \mathcal{D}'(\lambda, A, L, M, \mathcal{E}) \neq 0.$$

Which proves that $\lambda = (k\pi)^2$ is a simple eigenvalue.

Remark 13. In [17], it is proved that in the case of two identical branches, the eigenvalues are asymptotically given by $\lambda_{2k} = (k\pi)^2$ and $\lambda_{2k+1} = (\frac{\pi}{4} + k\pi)^2$. We see that the presence of a third branch gives an additional eigenvalue between the eigenvalues $\lambda_{2k} = (k\pi)^2$ and $\lambda_{2k+2} = ((k+1)\pi)^2$. Nevertheless this does not change the asymptotic behaviour of the spectral gap: $\lambda_{k+1} - \lambda_k = O(k)$.

3.2.2. What happens in the general case?

In both Theorems 7 and 9, some particular assumptions on $\sin(l_j a_j \sqrt{\lambda}) = 0$ have been put. It is now time to deal with the general case. Unfortunately, it is hard to deal with the case $\sin(l_j a_j \sqrt{\lambda}) = 0$ for some values of $j \in \{1, \dots, N\}$. The special case in the above Theorem 9 shows that λ^2 is an eigenvalue if $\sin(l_j a_j \sqrt{\lambda}) = 0$ for all values of $j \in \{1, \dots, N\}$. What happens if $\sin(l_j a_j \sqrt{\lambda}) = 0$ for some values of $j \in \{1, \dots, N\}$ but not for all of them?

A beginning of answer is given by the two following examples.

We consider two simple cases where it is possible to compute directly the characteristic equation due to the small number of branches using a formal calculation software. Let us describe these two examples.

Example 14 (*A chain of two branches supposed to be not identical and with the interior point mass equal to 1*). We set $l_1 a_1^{-1/4} = 1$, $l_2 a_2^{-1/4} = b$ and $M_1 = 1$. Assume that $\sin(\sqrt{\lambda}) = 0$ and $\sin(b\sqrt{\lambda}) \neq 0$.

Then the characteristic equation is

$$\mathcal{D}(\lambda) = \varepsilon \left(2 \cosh(b\sqrt{\lambda}) \sinh(\sqrt{\lambda}) + 2 \sinh(b\sqrt{\lambda}) \cosh(\sqrt{\lambda}) + m_1 \sqrt{\lambda} \sinh(b\sqrt{\lambda}) \sin(\sqrt{\lambda}) \right),$$

where $\varepsilon = \cos(\sqrt{\lambda}) \in \{-1, +1\}$.

It is clear that $\mathcal{D}(\lambda) \neq 0$, for all $b > 0$. Consequently we deduce that λ^2 is not an eigenvalue.

Example 15 (*A chain of three branches with two identical branches and with the interior point masses equal to 1*). We set $l_1 a_1^{-1/4} = l_2 a_2^{-1/4} = 1$, $l_3 a_3^{-1/4} = b$ and $M_1 = M_2 = 1$. Assume that $\sin(\sqrt{\lambda}) = 0$ and $\sin(b\sqrt{\lambda}) \neq 0$.

Then the characteristic equation is of the form

$$\mathcal{D}(\lambda) = \sin(b\sqrt{\lambda}) \psi_\lambda(b)$$

where ψ_λ is an analytic function which is too complicated to be given here.

A numerical analysis shows that the zeros of ψ_λ form a discrete set. For instance the zeros of ψ_λ in the interval $[0, 4]$ are approximatively

$$b = 0.360422, \quad b = 1.368071, \quad b = 2.368084.$$

So we deduce that λ^2 is not an eigenvalue except for some special values of b .

4. Asymptotic behaviour of the eigenvalues

This last section is devoted to the asymptotic behaviour of the eigenvalues of the operator \mathcal{A} defined in Section 2.2.

As it was announced in the introduction, we follow von Below [11] as Ali Mehmeti and Nicaise have done before ([1,31] and [22]) i.e. we establish the Weyl's formula with the help of the min–max principle of Courant–Weyl. The idea is to compare the eigenvalues of the operator \mathcal{A} with those of the same operator on each edge but with different boundary conditions, which are chosen so that the computation of the eigenvalues is easy.

4.1. Application of a corollary of the min–max principle of Courant–Weyl

Our aim is to apply Corollary 2.1.4 recalled in [1], which is a corollary of the min–max principle of Courant–Weyl. The exact formulation is:

Corollary 16 (*Corollary of the min–max principle of Courant–Weyl*). Let V , W and H be Hilbert spaces such that W is continuously embedded in V and densely and continuously embedded in H . Assume $a : V \times V \rightarrow \mathbb{C}$ is a continuous, symmetric, coercitive sesquilinear form. Let \mathcal{A}^W

and \mathcal{A}^V with corresponding eigenvalues μ_k^W and μ_k^V ($k \in \mathbb{N}$) be constructed from the triples (W, H, a) and (V, H, a) via Friedrichs Theorem. Then

$$\mu_k^V \leq \mu_k^W, \quad \forall k \in \mathbb{N}.$$

Notation 17. Let us recall

$$V = \left\{ U = (u, z) \in \prod_{j=1}^N H^2((0, l_j)) \times \mathbb{R}^n : \text{satisfying (2), (3), (5)} \right\}$$

is a Hilbert space endowed with the inner product

$$(U, W)_V = ((u, z), (w, s))_V = \sum_{j=1}^N (u_j, w_j)_{H^2((0, l_j))} + \sum_{i=1}^n z_i s_i,$$

where $(\cdot, \cdot)_{H^2((0, l_j))}$ is the usual inner product on $(0, l_j)$ and, for any $(U, W) \in V^2$

$$a(U, W) = \sum_{j=1}^N a_j \int_0^{l_j} u_{jx_j^{(2)}}(x_j) w_{jx_j^{(2)}}(x_j) dx_j.$$

Define

$$V_D = \left\{ U = (u, z) \in \prod_{j=1}^N H^2((0, l_j)) \times \mathbb{R}^n : \text{satisfying (2), (34), (5)} \right\}$$

with

$$\frac{\partial u_j}{\partial v_j}(E_i) = 0, \quad \forall j \in N_i, \quad \forall i \in I_{\text{int}} \quad (34)$$

and

$$V_N = \left\{ U = (u, z) \in \prod_{j=1}^N H^2((0, l_j)) \times \mathbb{R}^n : \text{satisfying (5)} \right\}.$$

For any $\epsilon > 0$ and $(U, W) = ((u, z), (w, s)) \in V_N^2$,

$$a_\epsilon(U, W) = \sum_{j=1}^N a_j \int_0^{l_j} u_{jx_j^{(2)}}(x_j) w_{jx_j^{(2)}}(x_j) dx_j + \epsilon \cdot \left(\sum_{j=1}^N (u_j, w_j)_{H^2((0, l_j))} + \sum_{i=1}^n z_i s_i \right).$$

Recall that conditions (2) to (6) are given in Section 2.2.

Proposition 18 (Properties of the spaces V_D , V and V_N and of the operators constructed from them).

1. $V_D \hookrightarrow V \hookrightarrow V_N \hookrightarrow H$ and V_D is dense in H .
2. $a : V \times V \rightarrow \mathbb{C}$, $a : V_D \times V_D \rightarrow \mathbb{C}$ and $a_\epsilon : V_N \times V_N \rightarrow \mathbb{C}$ are continuous, symmetric, coercitive sesquilinear forms.

3. The operators \mathcal{A}^{V_D} , \mathcal{A}^V and $(\mathcal{A}^{V_N} + \epsilon \cdot Id)$ constructed from the triples (V_D, H, a) , (V, H, a) and (V_N, H, a_ϵ) via Friedrichs Theorem are defined as follows:

$$\begin{cases} D(\mathcal{A}^{V_D}) = \{(u, z) \in H: z = 0, u_j \in H^4(0, l_j) \text{ satisfying (2), (34), (5), (6), (39)}\}, \\ \forall (u, z) \in D(\mathcal{A}^{V_D}), \quad \mathcal{A}^{V_D}(u, z) = (a_j u_{j x_j^{(4)}})_{j=1}^N, 0, \end{cases} \quad (35)$$

$$\begin{cases} D(\mathcal{A}^V) = \{(u, z) \in H: u_j \in H^4((0, l_j)) \text{ satisfying (2) to (6)}\}, \\ \forall (u, z) \in D(\mathcal{A}^V), \quad \mathcal{A}^V(u, z) = \left((a_j u_{j x_j^{(4)}})_{j=1}^N, -\frac{1}{M_i} \left(\sum_{j \in N_i} a_j \frac{\partial^3 u_j}{\partial v_j^3}(E_i) \right)_{i=1}^n \right), \end{cases} \quad (36)$$

$$\begin{cases} D(\mathcal{A}^{V_N}) = \{(u, z) \in H: u_j \in H^4(0, l_j) \text{ satisfying (5), (6), (38), (39)}\}, \\ \forall (u, z) \in D(\mathcal{A}^{V_N}), \quad \mathcal{A}^{V_N}(u, z) = ((a_j u_{j x_j^{(4)}})_{j=1}^N, 0), \end{cases} \quad (37)$$

with

$$\frac{\partial^2 u_j}{\partial v_j^2}(E_i) = 0, \quad \forall j \in N_i, \forall i \in I_{\text{int}}, \quad (38)$$

and

$$\frac{\partial^3 u_j}{\partial v_j^3}(E_i) = 0, \quad \forall j \in N_i, \forall i \in I_{\text{int}}. \quad (39)$$

4. Denoting by μ_k^D , μ_k and μ_k^N the eigenvalues of the operators \mathcal{A}^{V_D} , \mathcal{A}^V and \mathcal{A}^{V_N} , it holds

$$\mu_k^N \leq \mu_k \leq \mu_k^D, \quad \forall k \in \mathbb{N}. \quad (40)$$

Proof.

1. Note that (34) \Rightarrow (3). The embeddings are clearly continuous since the norms are identical and V_D contains $\prod_{j=1}^N C_c^\infty(0; l_j) \times \mathbb{R}^n$ which is dense in H .

2. The continuity and symmetry are clear. Lemma 2 in the proof of Lemma 1 (Section 2.2) was the technical tool to prove that $a : V \times V \rightarrow \mathbb{C}$ is coercitive. It still holds for $a : V_D \times V_D \rightarrow \mathbb{C}$. On the other hand (2) does not hold for the elements of V_N so Lemma 2 is not valid anymore. But the additional term with ϵ makes $a_\epsilon : V_N \times V_N \rightarrow \mathbb{C}$ coercitive.

3. The operators are constructed from the triples using the same technique as in the proof of Lemma 1 (Section 2.2) where \mathcal{A}^V was constructed and called \mathcal{A} .

4. Corollary 13 is applied to the operators \mathcal{A}^{V_D} , \mathcal{A}^V and $(\mathcal{A}^{V_N} + \epsilon \cdot Id)$. The eigenvalues of the latter operator are the $(\mu_k^N + \epsilon)$'s. Taking the limit as ϵ tends to zero, we deduce the results. \square

4.2. Weyl's formula

The asymptotic behaviour of the eigenvalues of the operator \mathcal{A} will now be deduced from the estimates given in Proposition 15.

Lemma 19 (Computation of the eigenvalues of \mathcal{A}^{V_D} and \mathcal{A}^{V_N}). Denoting by J_{ext} (respectively J_{int}) the set of indices of the exterior (respectively interior) edges i.e.

$$J_{\text{ext}} = \{j: \exists i \in I_{\text{ext}}, j \in N_i\} \quad \text{and} \quad J_{\text{int}} = \{1, \dots, N\} \setminus J_{\text{ext}}$$

the set of the eigenvalues of

- \mathcal{A}^{VD} is: $\bigcup_{k \in \mathbb{N}} [\bigcup_{j \in \{1, \dots, N\}} v_{jk}^N]$,
- \mathcal{A}^{VN} is: $\{0\} \cup [\bigcup_{k \in \mathbb{N}} [\bigcup_{j \in \{1, \dots, N\}} v_{jk}^N]]$,

where $v_{jk}^N = (\frac{k\pi}{b_j} + O(1))^4$ as k tends to infinity.

Note that an edge is called exterior if it contains an exterior node and it is called interior otherwise. Since the graph is assumed to be connected, an exterior edge contains only one exterior node. Recall that the definitions of I_{int} and N_i are given in Section 2.1 and $b_j = l_j \cdot a_j^{-1/4}$ with a_j and l_j defined in Section 2.2.

Proof. The ideas are completely analogous to those of Dekoninck and Nicaise [22], i.e. to get the eigenvalues of \mathcal{A}^{VD} , it is sufficient to compute the eigenvalues of two problems:

1. The Dirichlet problem on an exterior branch $(0, l_j)$ ($j \in J_{\text{ext}}$):

$$\begin{cases} a_j u_{jx_j^{(4)}} = \mu_{jk}^D u & \text{on } (0, l_j), \\ u(0) = u''(0) = u(l_j) = u'(l_j) = 0. \end{cases}$$

Here 0 is supposed to correspond to the exterior node and l_j to the interior one.

First of all 0 is not an eigenvalue. Then a classical computation using the fundamental system of Lemma 5 leads to the other eigenvalues: a nonvanishing eigenvalue μ_j^N of this problem is given by $\mu_j^N = a_j l_j^{-1/4} \alpha^2 = b_j^{-1/4} \alpha^2$ where α is a root of $\cosh(\sqrt{\alpha}) \sin(\sqrt{\alpha}) - \cos(\sqrt{\alpha}) \sinh(\sqrt{\alpha}) = 0$.

So the roots denoted by α_k ($k \in \mathbb{N}$) behave like those of $\sin(\sqrt{\beta}) = \cos(\sqrt{\beta})$ as k tends to infinity since $\sinh(x)/\cosh(x)$ tends to 1 if x tends to $+\infty$. Now $\beta_k = (\frac{\pi}{4} + k\pi)^2$ for all $k \in \mathbb{N}$.

2. The Dirichlet problem on an interior branch $(0, l_j)$ ($j \in J_{\text{int}}$):

$$\begin{cases} a_j u_{jx_j^{(4)}} = \mu_{jk}^D u & \text{on } (0, l_j), \\ u'(0) = u(0) = u'(l_j) = u(l_j) = 0. \end{cases}$$

Here again 0 corresponds to the exterior node and l_j to the interior one.

First of all 0 is not an eigenvalue. Then a classical computation using the fundamental system of Lemma 5 leads to the other eigenvalues: a nonvanishing eigenvalue μ_j^N of this problem is given by $\mu_j^N = a_j l_j^{-1/4} \alpha^2 = b_j^{-1/4} \alpha^2$ where α is a root of $\cosh(\sqrt{\alpha}) \cos(\sqrt{\alpha}) - 1 = 0$.

The roots denoted by α_k ($k \in \mathbb{N}$) behave like those of $\cos(\sqrt{\gamma}) = 0$ as k tends to infinity since $1/\cosh(x)$ tends to 0 if x tends to $+\infty$. Now $\gamma_k = (\frac{\pi}{2} + k\pi)^2$ for all $k \in \mathbb{N}$.

Likewise for the eigenvalues of \mathcal{A}^{VN} , we compute the eigenvalues of two problems:

1. The Neumann problem on an exterior branch $(0, l_j)$ ($j \in J_{\text{ext}}$):

$$\begin{cases} a_j u_{jx_j^{(4)}} = \mu_{jk}^N u & \text{on } (0, l_j), \\ u(0) = u''(0) = u(l_j) = u''(l_j) = 0. \end{cases}$$

First of all, 0 is an eigenvalue of this problem with multiplicity two.

And a nonvanishing eigenvalue μ_j^N of this problem is given by $\mu_j^N = a_j l_j^{-1/4} \alpha^2 = b_j^{-1/4} \alpha^2$ where α is a root of $\cosh(\sqrt{\alpha}) \sin(\sqrt{\alpha}) - \cos(\sqrt{\alpha}) \sinh(\sqrt{\alpha}) = 0$. So, except for 0, the eigenvalues are the same as for \mathcal{A}^{VD} .

2. The Neumann problem on an interior branch $(0, l_j)$ ($j \in J_{\text{int}}$):

$$\begin{cases} a_j u_{jx_j^{(4)}} = \mu_{jk}^N & \text{on } (0, l_j), \\ u(0) = u''(0) = u'''(l_j) = u''(l_j) = 0. \end{cases}$$

Here again 0 corresponds to the exterior node and l_j to the interior one and 0 is an eigenvalue of this problem with multiplicity two.

And a nonvanishing eigenvalue μ_j^N of this problem is given by $\mu_j^N = a_j l_j^{-1/4} \alpha^2 = b_j^{-1/4} \alpha^2$ where α is a root of $\cosh(\sqrt{\alpha}) \cos(\sqrt{\alpha}) - 1 = 0$.

So, except for 0, the eigenvalues are the same as for \mathcal{A}^{V_D} .

So all the eigenvalues have the same asymptotic behaviour given by the above lemma. \square

Note that 0 is an eigenvalue of \mathcal{A}^{V_N} which is in accordance with the fact that the sesquilinear form a introduced in the proof of Lemma 1 (Section 2.2) is not coercitive on V_N . That is why a_ϵ had to be introduced in Notation 14 to apply Friedrichs Theorem.

Theorem 20 (Eigenvalue asymptotics). *Let \mathcal{A} be the nonnegative self-adjoint operator defined in Section 2.2 and denote by $\{\mu_k\}_{k \in \mathbb{N}}$ the monotonically increasing sequence of the eigenvalues of \mathcal{A} (repeated according to their multiplicities) then it holds:*

$$\lim_{k \rightarrow \infty} \frac{\mu_k}{k^4} = \pi^4 \left(\sum_{j=1}^N l_j a_j^{-\frac{1}{4}} \right)^{-4}.$$

(Note that the eigenvalues of the operator \mathcal{A} , which were denoted by λ^2 in Sections 2 and 3, are called μ from now on.)

Proof. According to Lemma 16, the eigenvalues of \mathcal{A}^{V_D} and \mathcal{A}^{V_N} have the same asymptotic behaviour as k tends to infinity:

$$\mu_{jk}^D = \left(\frac{k\pi}{b_j} + O(1) \right)^4 \quad \text{and} \quad \mu_{jk}^N = \left(\frac{k\pi}{b_j} + O(1) \right)^4. \quad (41)$$

Following Dekoninck and Nicaise (Section 4 of [22]), we will apply the following lemma:

Lemma 21. *Let $\{\mu_k\}_{k \in \mathbb{N}}$ be an increasing sequence of nonnegative real numbers and denote by $N(r)$ the number of μ_k in $[0, r]$. Let c and l be two fixed elements of \mathbb{R}^{+*} . Then*

$$\mu_k = (ck + O(1))^l \iff N(r) = c^{-1} r^{\frac{1}{l}} + O(1).$$

Applying this lemma to the sequence $\{\mu_{jk}^D\}_{k \in \mathbb{N}}$ with fixed j and $c = \frac{\pi}{b_j}$, $l = 4$, (41) is equivalent to

$$N_j^D(r) = \frac{b_j}{\pi} \cdot r^{\frac{1}{4}} + O(1), \quad (42)$$

where $N_j^D(r)$ is the number of μ_{jk}^D in $[0, r]$. Now denoting by $N^D(r)$ the number of μ_k^D in $[0, r]$

$$N^D(r) = \sum_{j=1}^N N_j^D(r) = \left(\sum_{j=1}^N b_j \right) \frac{r^{1/4}}{\pi} + O(1).$$

Table 1
 $b_1 = 0.8, b_2 = 1.2, M_1 = 1.0$

k	λ_k^N	λ_k	λ_k^D
0	0	1.78091	10.7071
1	0	9.05642	24.0909
2	0	20.4658	34.6978
3	0	34.0494	72.3942
4	10.7071	61.685	78.0702
5	24.0909	78.2888	123.798
6	34.6978	116.525	162.887
7	72.3942	151.44	188.91
8	78.0702	184.931	267.73
9	123.798	246.74	278.546
10	162.887	276.447	360.258

Table 2
 $b_1 = 0.8, b_2 = 1.0, b_3 = 1.2, M_1 = 1.0, M_2 = 1.5$

k	λ_k^N	λ_k	λ_k^D
0	0	0.775571	10.7071
1	0	3.26419	22.3733
2	0	8.90296	31.4657
3	0	16.2583	34.698
4	10.7071	27.4522	61.6728
5	22.3733	33.5621	72.3943
6	31.4657	52.026	101.969
7	34.698	68.9443	120.903
8	61.6728	88.5291	123.798
9	72.3943	108.145	188.91
10	101.969	123.349	199.859

Thus

$$\mu_k^D = \left(\frac{k\pi}{\sum_{j=1}^N b_j} + O(1) \right)^4 \quad \text{and} \quad \mu_k^N = \left(\frac{k\pi}{\sum_{j=1}^N b_j} + O(1) \right)^4.$$

The result now follows from the estimates (40) since $b_j = l_j a_j^{-1/4}$. \square

Some numerical values for the squares of the eigenvalues λ_k^N , λ_k and λ_k^D are given in Tables 1 and 2 for two chains of two and three different branches, respectively. They confirm the estimates of Proposition 15. Note that $\mu_k = \lambda_k^2$.

Acknowledgments

The authors are grateful to Professor F. Ali Mehmeti and Professor S. Nicaise for fruitful discussions and remarks.

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